



Price uncertainty and the existence of financial equilibrium

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Abstract

We consider a pure exchange economy, with incomplete financial markets, where agents face an ‘exogenous uncertainty’, on the future state of nature, and an ‘endogenous uncertainty’, on the future price in each random state. Namely, every agent forms price anticipations on each future spot market, distributed along an idiosyncratic probability law. At a sequential equilibrium, all agents expect the ‘true’ price as a possible outcome and elect optimal strategies at the first period, which clear on all markets at every time period. We show that, provided the endogenous uncertainty is large enough, a sequential equilibrium exists under standard conditions, for all types of financial structures (i.e., with real, nominal and mixed assets). This result suggests that standard existence problems of sequential equilibrium models, following Hart (1975), stem from the single price expectation assumption.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, expectations, incomplete markets, asymmetric information, arbitrage, existence proof.

JEL Classification: D52

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1 Introduction

The traditional approach to sequential equilibrium relies on the assumption, introduced by Radner (1972), that agents have a ‘*perfect foresight*’ of future prices. That is, agents anticipate with certainty exactly one price for each commodity (or asset) in each random state, which turns out to be the true price, if that state prevails tomorrow. At a sequential equilibrium, agents elect optimal decisions at the first period, which clear on all markets *ex post*. A convenient outcome of perfect foresight is that equilibrium unfolds sequentially: after all uncertainty is removed, agents never face bankruptcy, or need trade again to optimize their welfare, and all markets clear at agents’ *ex ante* decisions. Yet, the perfect foresight assumption is a sufficient but, by no means, a necessary condition to guarantee these outcomes. As argued below, the perfect foresight equilibrium is a restrictive notion of *sequential* (as opposed to *temporary*) equilibrium, resulting in several intricate problems.

The traditional criticism of the perfect foresight model is that, quoting Radner himself (1982), “*it requires of the traders a capacity for imagination and computation far beyond what is realistic*”. Indeed, all traders need coordinate on the same expectation, and forecast the future price exactly and solely, on every spot market. Typically, if only one agent deviated privately from the common price anticipation, all rational agents, in Radner’s sense, would turn out to have mistaken forecasts *ex post*. This typical ‘*common knowledge*’ requirement (of the price forecast) is not the only problem that the perfect foresight model embeds. This model would not explain the effects of agents’ (possibly irrational or mistaken) beliefs on equilibrium prices and quantities traded on markets. In particular, it would not explain speculation, bubbles and crashes, observed on actual markets, as sequential equilibrium phenomena. In some financial economies where real assets are sold short

unrestrainedly, a perfect foresight equilibrium may even fail to exist, as shown by the examples of Hart (1975), Momi (2000), Busch-Govindan (2004), among others.

In general equilibrium theory, the perfect foresight model has remained, to our knowledge, the only setting to deal with *sequential* equilibrium so far. Yet, all above mentioned problems of the perfect foresight model might be solved by introducing an alternative concept of sequential equilibrium, which lets agents be uncertain of future prices, but cautious enough never to need revise their forecasts ex post. We refer to the latter concept as the ‘*correct foresight equilibrium*’ (C.F.E.), which is reached when agents have anticipations, which embed tomorrow’s ‘*true*’ price as a possible outcome, and make optimal decisions today, which clear markets and remain optimal ex post. In a companion paper, we show, on an example, that the C.F.E. model indeed explains volatility, speculation and crashes on incomplete financial markets at a sequential equilibrium, stemming from agents’ anticipations when no change in the fundamentals takes place. Hereafter, we focus on existence issues and prove, on a simple model with price uncertainty, that a sequential equilibrium always exists under standard conditions, whatever the financial structure (i.e., with real, nominal or mixed assets), to the difference of Hart and alii.

The model we propose is a two-period pure exchange economy, where agents face an exogenous uncertainty, represented by finitely many random states of nature, exchange goods on spot markets, for the purpose of consumption, and trade, unrestrainedly, on (incomplete) financial markets, so as to transfer wealth across periods and states. At the first period, besides the above exogenous uncertainty, agents face an ‘*endogenous uncertainty*’ on each future spot price. Namely, consumers have a (private) set of plausible spot prices for each future state, called expectations, distributed along an idiosyncratic probability law, which we call a belief. The latter

uncertainty is traditionally referred to as ‘*endogenous*’ because it both affects and is focussed on the endogenous price variable. As our main Theorem, we show that, if the support of beliefs is large enough, the existence of a C.F.E. is still guaranteed by standard assumptions, on any financial market.

To simplify exposition, we restrict the model to symmetric supports of beliefs, that is, to agents having a common set of expected states and prices. This restriction may, indeed, be dropped without changing our Theorem, but it would lead to much longer developments, since agents might need refine their anticipations first, until their beliefs became consistent, in the sense that no trader had an arbitrage opportunity, stemming from some other traders’ (mistaken) anticipations (see [4]). With identical supports, the existence of equilibrium is guaranteed without any prior refinement of beliefs, which permits to skip that step.

It must be stressed that our equilibrium notion is, indeed, a *sequential* one, that is, differs from the *temporary* equilibrium’s, as introduced by Hicks (1939) and developed, later, by Grandmont (1977, 1982), Green (1973), Hammond (1983), Balasko (2003), among others, explaining why we do not extend on this literature. At a temporary equilibrium, agents would typically revise their plans and forecasts *ex post*. Indeed, only current markets need clear at agents’ equilibrium decisions and the true price needs not be anticipated by every trader correctly, which may lead, tomorrow, to bankruptcy situations and new welfare improving exchanges. None of these outcomes, antonymous to sequential unfolding, occurs at a C.F.E.

The paper is organized as follows: Section 2 presents the model and its basic concepts; Section 3 states and proves the Theorem; an Appendix proves Lemmas.

2 The model

We consider a pure-exchange financial economy with two time-periods ($t \in \{0, 1\}$) and two markets, a commodity market and a financial market. There is an a priori uncertainty at the first period ($t = 0$) about which state s of a given state space S will prevail at the second period ($t = 1$), when all uncertainty is removed. The state of nature at $t = 0$ is non random and denoted by $s = 0$. The sets of agents (or consumers), $I := \{1, \dots, m\}$, of commodities, $\mathcal{L} := \{1, \dots, L\}$, of states of nature, $S := \{1, \dots, N\}$, and financial assets, $\mathcal{J} := \{1, \dots, J\}$, are all finite.

Before presenting the model, we introduce notations, which are used throughout.

2.1 The model's notations

Throughout, we denote by \cdot and $\|\cdot\|$, respectively, the scalar product and Euclidean norm on an Euclidean space, and by $\mathcal{P}(K)$ and $\mathcal{B}(K)$, respectively, the set of non-empty subsets and Borel sigma-algebra of a set or topological space, K . We let $s = 0$ be the non-random state at $t = 0$ and $S' := \{0\} \cup S$. For all sets, $\Sigma \in \{S, S'\}$, $A \in \mathcal{P}(\mathbb{R}^\Sigma)$, $B \in \mathcal{P}((\mathbb{R}^L)^\Sigma)$, $\Sigma \times J$ matrix, V (where $J \in \mathbb{N}^*$), tuple, $(\varepsilon, s, l, x, x', y, y') \in \mathbb{R}_{++} \times \Sigma \times \mathcal{L} \times \mathbb{R}^\Sigma \times \mathbb{R}^\Sigma \times (\mathbb{R}^L)^\Sigma \times (\mathbb{R}^L)^\Sigma$, we denote by:

- $A(s) \in \mathcal{P}(\mathbb{R})$, $B(s) \in \mathcal{P}(\mathbb{R}^L)$, $V(s) \in \mathbb{R}^J$, $x(s) \in \mathbb{R}$, $y(s) \in \mathbb{R}^L$, respectively, the sets of projections, row, scalar, vector, indexed by $s \in \Sigma$, of A , B , V , x , y ; and we also denote $A_s := A(s)$, $B_s := B(s)$, $x_s := x(s)$ and $y_s := y(s)$;
- $y^l(s) := y_s^l$ the l^{th} component of $y(s) \in \mathbb{R}^L$;
- $x \leq x'$ and $y \leq y'$ (resp. $x << x'$ & $y << y'$) the relations $x(s) \leq x'(s)$ and $y^l(s) \leq y^l(s)$ (resp. $x(s) < x'(s)$ & $y^l(s) < y^l(s)$) for all $(l, s) \in \{1, \dots, L\} \times \Sigma$;
- $x < x'$ (resp. $y < y'$) the joint relations $x \leq x'$, $x \neq x'$ (resp. $y \leq y'$, $y \neq y'$);

- $\mathbb{R}^{L\Sigma} := (\mathbb{R}^L)^\Sigma$, $\mathbb{R}_+^{L\Sigma} = \{x \in \mathbb{R}^{L\Sigma} : x \geq 0\}$ and $\mathbb{R}_+^\Sigma := \{x \in \mathbb{R}^\Sigma : x \geq 0\}$,
 $\mathbb{R}_{++}^{L\Sigma} := \{x \in \mathbb{R}^{L\Sigma} : x \gg 0\}$ and $\mathbb{R}_{++}^\Sigma := \{x \in \mathbb{R}^\Sigma : x \gg 0\}$,
- $\mathcal{M}_0 := \{[p_0, q] \in \mathbb{R}_+^L \times \mathbb{R}^J : \|p_0\| + \|q\| = 1\}$;
- $\mathcal{M}_s := \{[s, p_s] : p_s \in \mathbb{R}_+^L, \|p_s\| = 1\}$ and $\mathcal{M}_s^\varepsilon := \{[s, p_s] \in \mathcal{M}_s : p_0 \in [\varepsilon, 1]^L\}$, for $s \in S$;
- $\mathcal{M} := \Pi_{s \in S'} \mathcal{M}_s$, $\mathcal{M}^\varepsilon := \mathcal{M}_0 \times \Pi_{s \in S} \mathcal{M}_s^\varepsilon$, $\Omega := \cup_{s \in S} \mathcal{M}_s$ and $\Omega^\varepsilon := \cup_{s \in S} \mathcal{M}_s^\varepsilon$.

2.2 The commodity and asset markets

The L commodities, $l \in \mathcal{L}$, are used for the purpose of consumption and may be exchanged between agents on spot markets. There are $\#S'$ ex ante possible spot markets, namely one in each state $s \in S'$. In each future state, $s \in S$, an expectation of the spot price, $p_s \in \mathbb{R}_+^L$, is denoted by $\omega_s := [s, p_s] \in S \times \mathbb{R}_+^L$. At little cost, we normalize admissible expectations in each state $s \in S$ to the above set \mathcal{M}_s .

Agents exchange commodities in order to increase their welfare. Trade may take place because each agent, $i \in I$, can rely on an endowment, $e_i := (e_i(s)) \in \mathbb{R}_{++}^{LS'}$, of the L goods, which grants her the commodity bundle $e_i(0) \in \mathbb{R}_{++}^L$ at $t = 0$, and $e_i(s) \in \mathbb{R}_{++}^L$, in each state $s \in S$ if this state prevails at $t = 1$. Ex post, the agent's welfare is measured by $u_i(x_0, x_1) \in \mathbb{R}_+$, where $x_0 := (x_0^1, \dots, x_0^L) \in \mathbb{R}_+^L$ and $x_1 := (x_1^1, \dots, x_1^L) \in \mathbb{R}_+^L$ are the vectors of consumptions, respectively, at $t = 0$ and $t = 1$, and $u_i : \mathbb{R}_+^{2L} \rightarrow \mathbb{R}_+$ is a utility function, assumed to be C^1 and strictly increasing.

At this stage, we may state a Lemma, which will permit, later, to restrict admissible prices and price expectations, to some set \mathcal{M}^ε for $\varepsilon > 0$.

Lemma 1 *Let $e := \max_{(s,l) \in S' \times \mathcal{L}} \sum_{i=1}^m e_i^l(s)$ and $\beta := \sup \frac{\partial u_i}{\partial y^l}(x, y) / \frac{\partial u_i}{\partial y^l}(x, y)$, for $i \in I$, $(l, l') \in \mathcal{L}^2$, $(x, y) \in ([0, e]^L)^2$. Then, $(e, \beta) \in \mathbb{R}_{++}^2$ and we denote $\varepsilon_0 := \frac{1}{\beta \sqrt{L}} \leq \frac{1}{\sqrt{L}}$.*

Proof The proof is immediate from the fact that, for each $i \in I$, $e_i \gg 0$ and u_i is C^1 and strictly increasing on the compact set $[0, e]^{2L}$. \square

The financial market permits limited transfers across periods and states, via J assets, also called securities, $j \in \mathcal{J} := \{1, \dots, J\}$, which are exchanged at $t = 0$ and pay off at $t = 1$. Assets may be nominal or real (i.e., pay off in account units or in commodities). For any expectation, $p := ([s, p_s]) \in \Pi_{s \in S} \mathcal{M}_s$, the payoffs, $v_j([s, p_s]) \in \mathbb{R}$, of each asset $j \in \{1, \dots, J\}$ in each state $s \in S$, define a fixed $S \times J$ -matrix, $V(p) = (v_j([s, p_s]))$, referred to as the payoff matrix, and such that $V : p \in \mathcal{M} \mapsto V(p)$ is continuous, from the definition. For every $s \in S$ and every $p := ([s, p_s]) \in \Pi_{s \in S} \mathcal{M}_s$, we shall denote by $V(p)(s) := V([s, p_s]) := (v_j([s, p_s]))_{j \in \mathcal{J}} \in \mathbb{R}^J$ the s^{th} -row of $V(p)$.

Provided she can afford, every agent $i \in I$ may take unrestrained positions, $z_i^j \in \mathbb{R}$ (positive, if purchased; negative, if sold), in every security $j \in \{1, \dots, J\}$, which define her portfolio, $z_i := (z_i^j) \in \mathbb{R}^J$. When an asset price, $q \in \mathbb{R}^J$, is observed at $t = 0$, a portfolio, $z \in \mathbb{R}^J$, is thus a contract, which costs $q \cdot z$ units of account at $t = 0$, and promises to pay $V([s, p_s]) \cdot z$ units, in each state $s \in S$, for every expectation $[s, p_s] \in \mathcal{M}_s$, if p_s obtains. Similarly, we henceforth normalize first period prices, $\omega_0 := (p_0, q)$, to the set \mathcal{M}_0 of sub-Section 2.1.

2.3 Information and beliefs

At $t = 0$, agents form private anticipations of future spot prices, distributed along idiosyncratic probability laws, which represent their endogenous uncertainty.

Definition 1 *For every probability, π , on $(\Omega, \mathcal{B}(\Omega))$, scalar, $\varepsilon \in \mathbb{R}_{++}$, expectation, $\omega_s := [s, p] \in \Omega$, we let $B(\omega_s, \varepsilon) := \{[\bar{s}, \bar{p}] \in \Omega : \|\bar{p} - p\| + |\bar{s} - s| < \varepsilon\}$ be an open ball and denote by $P(\pi) := \{\omega \in \Omega : \pi(B(\omega, \varepsilon)) > 0, \forall \varepsilon > 0\}$ the support of π , a compact set. A probability, π , on $(\Omega, \mathcal{B}(\Omega))$, is called a belief if the following Condition holds:*

(a) $\exists \varepsilon \in \mathbb{R}_{++} : P(\pi) \subset \Omega^\varepsilon$.

We denote by \mathcal{B} the set of all beliefs. Let $C_0(\Omega, \mathbb{R})$ be the set of continuous mappings from Ω to \mathbb{R} . A collection, $\Pi := (\Pi_i)$, of m mappings, $\Pi_i : \mathcal{M}_0 \rightarrow \mathcal{B}$ (for $i \in I$), is said to be an expectation structure if, for each $i \in I$, the following Conditions hold:

(b) $[\omega_0 \in \mathcal{M}_0, (\omega_0^n)_{n \in \mathbb{N}} \in \mathcal{M}_0^{\mathbb{N}}, \omega_0 = \lim_{n \rightarrow \infty} \omega_0^n, f \in C_0(\Omega, \mathbb{R})] \Rightarrow [\int_{\Omega} f d(\Pi_i(\omega_0)) = \lim_{n \rightarrow \infty} \int_{\Omega} f d(\Pi_i(\omega_0^n))];$

(c) $\forall (\omega_0, \omega'_0) \in \mathcal{M}_0^2, P(\Pi_i(\omega_0)) = P(\Pi_i(\omega'_0)) := P \subset \Omega;$

(d) $\Omega^{\varepsilon_0} \subset P$, along Condition (c) and Lemma 1 above.

We denote by $\mathcal{ES} \subset (\mathcal{B}^{\mathcal{M}_0})^m$ the set of all expectation structures.

Remark 1 Without changing the paper's results, a belief could be defined as a probability on $S \times \mathbb{R}_+^L$, whose support cannot take indefinitely large or low values. Normalized expectations, here, simplify presentation, but do not reduce generality.

Remark 2 Condition (b) is a standard continuity condition on a probability space.

Remark 3 Condition (c) states that observing first period prices does not affect the support of agents' beliefs, which are, moreover, symmetric. As explained in the introduction, our results would still hold with asymmetric (and varying) supports.

Remark 4 Condition (d) states that agents are cautious enough not to rule out completely any expectation of the (true) spot prices, out of a small neighborhood of zero, namely, $\Omega \setminus \Omega^{\varepsilon_0}$. The fixed set, Ω^{ε_0} , could be narrowed down to what we call '*the minimum uncertainty set*', which is the set of all possible equilibrium prices tomorrow, given today's, when agents' beliefs are private and vary. If Condition (d) were reduced to the requirement that agents' expectations always contained that minimum uncertainty set, the paper's existence result could still be proved (even under agents' asymmetric information & beliefs), at the cost of long complex developments. In a seminal paper, we prefer to keep the uncertainty set large and

fixed, namely, Ω^{ε_0} , in order to prove and explain this existence result simply, based on the fact that all possible equilibrium prices need be in Ω^{ε_0} tomorrow (see below).

Henceforth, we assume that agents are endowed with an (arbitrary) expectation structure, $\Pi := (\Pi_i) \in \mathcal{ES}$, which is fixed and always referred to, unless stated otherwise, and we let $P := P(\Pi_i(\omega_0))$, and $\pi_i^{\omega_0} := \Pi_i(\omega_0)$, for every pair $(i, \omega_0) \in I \times \mathcal{M}_0$.

2.4 Consumers' behavior and the notion of equilibrium

In this sub-Section, we assume agents observe the market price at $t = 0$, namely, $\omega_0 := (p_0, q) \in \mathcal{M}_0$, which is set as given, when they make their trade and consumption plans. The generic i^{th} agent's set of consumption plans is defined as:

$$Y_i := \mathcal{C}_i(P, \mathbb{R}_+^{2L}),$$

where $\mathcal{C}_i(P, \mathbb{R}_+^{2L})$ stands for the set of consumption mappings, $y : P \rightarrow \mathbb{R}_+^L \times \mathbb{R}_+^L$, such that the first period consumption, $y(0) \in \mathbb{R}_+^L$, is constant (in $\omega \in P$) and the second period consumption plan, $\omega \in P \mapsto y(\omega) \in \mathbb{R}_+^L$, makes the welfare mapping, $\omega \in P \mapsto u_i(y(0), y(\omega)) \in \mathbb{R}_+$, continuous. The plan $y \in \mathcal{C}_i(P, \mathbb{R}_+^{2L})$, relates every spot price, $\omega := [s, p_s] \in P$ to a consumption decision, $y([s, p_s]) \in \mathbb{R}_+^L$, for $t = 1$, which is conditional on the joint conditions that state s prevailed and price $p_s \in \mathbb{R}_+^L$ be observed tomorrow. The continuity condition is a rationality rule, stating that no agent elects joint consumptions, whose utilities vary discontinuously with expectations.

Each agent $i \in I$ elects and implements a consumption and investment decision, or strategy, $[y, z] \in Y_i \times \mathbb{R}^J$, that she can afford on markets, given her endowment, $e_i \in \mathbb{R}_+^{L'}$, and her expectation set, P . This defines her budget set as follows:

$$B_i(\omega_0) := \{[y, z] \in Y_i \times \mathbb{R}^J : p_0 \cdot (y(0) - e_i(0)) \leq -q \cdot z, \quad p_s \cdot (y([s, p_s]) - e_i(s)) \leq V([s, p_s]) \cdot z, \forall [s, p_s] \in P\}.$$

An allocation, $(y_i) \in Y := \Pi_{i=1}^m Y_i$, is a collection of consumption plans across consumers. For all price collection, $(\omega_0, p := ([s, p_s])) \in \mathcal{M}$, we define the following sets of attainable allocations, portfolios and strategies, respectively:

$$\mathcal{A}(p) := \{(y_i) \in Y : \sum_{i=1}^m (y_i(0) - e_i(0)) = 0; \forall s \in S, \text{ such that } [s, p_s] \in P, \sum_{i=1}^m (y_i([s, p_s]) - e_i(s)) = 0\};$$

$$\mathcal{Z} := \{(z_i) \in (\mathbb{R}^J)^m : \sum_{i=1}^m z_i = 0\};$$

$$\mathcal{Y}(\omega_0, p) := \{([y_i, z_i]) \in \Pi_{i=1}^m B_i(\omega_0) : (y_i) \in \mathcal{A}(p), (z_i) \in \mathcal{Z}\}.$$

For every $\omega_0 \in \mathcal{M}_0$, we assume that each agent $i \in I$, given the first period price ω_0 , has preferences represented by the V.N.M. utility function (recalling that $\pi_i^{\omega_0} := \Pi_i(\omega_0) \in \mathcal{B}$ is a belief):

$$u_i^{\omega_0} : y \in Y_i \mapsto u_i^{\omega_0}(y) := \int_{\omega \in P} u_i(y(0), y(\omega)) d\pi_i^{\omega_0}(\omega).$$

The generic i^{th} agent's behavior is, then, to elect a strategy, which maximises this utility function in the buget set, that is, a strategy in $B_i^*(\omega_0) := \arg \max_{[y, z] \in B_i(\omega_0)} u_i^{\omega_0}(y)$.

The above economy, for a given expectation structure, $\Pi := (\Pi_i) \in \mathcal{ES}$, is denoted by \mathcal{E}_Π . An equilibrium of this economy is a collection of market prices, $(\omega_s) \in \Pi_{s \in S'} \mathcal{M}_s$ and optimal attainable strategies. Formally:

Definition 2 *A collection of prices, $(\omega_0 := [p_0, q], p := ([s, p_s])) \in \mathcal{M}$, and strategies, $([y_i, z_i]) \in \Pi_{i=1}^m B_i(\omega_0)$, defines a sequential equilibrium of the economy \mathcal{E}_Π , or correct foresight equilibrium (C.F.E.), (respectively, a temporary equilibrium of \mathcal{E}_Π) if the following Conditions (a)-(b)-(c)-(d) (respectively, Conditions (b)-(c)-(d)) hold:*

- (a) $\forall s \in S, [s, p_s] \in P$;
- (b) $\forall i \in I, [y_i, z_i] \in B_i^*(\omega_0) := \arg \max_{[y, z] \in B_i(\omega_0)} u_i^{\omega_0}(y)$;
- (c) $(y_i) \in \mathcal{A}(p)$;
- (d) $(z_i) \in \mathcal{Z}$.

The economy \mathcal{E}_Π is called “*standard*” under the two following Assumptions:

- **Assumption A1:** $\forall i \in I, e_i >> 0$;
- **Assumption A2:** $\forall i \in I, u_i$ is class C^1 , concave, and strictly increasing, i.e.,
 $\frac{\partial u_i}{\partial y^l}(y) > 0, \forall (y, l) \in \mathbb{R}_+^{2L} \times \{1, \dots, 2L\}$.

We can now state our main Theorem, which we prove in Section 3.

Theorem 1 *For every $\Pi := (\Pi_i) \in \mathcal{ES}$, a standard economy, \mathcal{E}_Π , admits a C.F.E.*

3 The existence proof

Throughout, $\Pi := (\Pi_i) \in \mathcal{ES}$ is set as given and the bounds of Lemma 1, e and ε_0 , the beliefs, $\pi_i^{\omega_0} := \Pi_i(\omega_0) \in \mathcal{B}$ and their common support (along Definition 1-(b)), $P := P(\pi_i^{\omega_0}) \subset \Omega$, for every $(i, \omega_0) \in I \times \mathcal{M}_0$, are always referred to.

The proof’s principle is to construct a sequence of auxiliary economies with finite expectations sets, each of which admits an equilibrium along Theorem 1 of [5], and to derive from a sequence of equilibria in auxiliary economies an equilibrium of the initial economy, \mathcal{E}_Π . To that aim, we need introduce auxiliary sets, in a first step.

3.1 Auxiliary sets

We first divide P in finer and finer partitions, letting for each $n \in \mathbb{N}$:

$$K_n := \{k_n := (k_n^1, \dots, k_n^L) \in (\mathbb{N} \cap [0, 2^n - 1])^L\};$$

$$P_{(s, k_n)} := P \cap (\{s\} \times \Pi_{l \in \{1, \dots, L\}}] \frac{k_n^l}{2^n}, \frac{k_n^l + 1}{2^n}]), \text{ for every } (s, k_n := (k_n^1, \dots, k_n^L)) \in S \times K_n.$$

Then, for each $(s, n, k_n) \in S \times \mathbb{N} \times K_n$, such that $P_{(s, k_n)} \neq \emptyset$, we select one element $g_{(s, k_n)}^n \in P_{(s, k_n)}$, define the set $G^n := \{g_{(\bar{s}, \bar{k}_n)}^n : \bar{s} \in S, \bar{k}_n \in K_n, P_{(\bar{s}, \bar{k}_n)} \neq \emptyset\}$ accordingly, and a so-called ‘equilibrium price’, $(\omega_0^n, p^n) \in \mathcal{M}^{\varepsilon_0}$, by induction, as follows:

- for $n = 0$, we let an arbitrary price, $(\omega_0^0, p^0) \in \mathcal{M}^{\varepsilon_0}$, be given, we select one price $g_{(s,0)}^0 \in P_{(s,0)} \neq \emptyset$, for each $s \in S$, and define $G^0 := \{g_{(s,0)}^0 : s \in S\}$ accordingly;
- for $n \in \mathbb{N}^*$ arbitrary, we assume the equilibrium price $(\omega_0^{n-1}, p^{n-1}) \in \mathcal{M}^{\varepsilon_0}$ and the set $G^{n-1} := \{g_{(s,k_{n-1})}^{n-1} : s \in S, k_{n-1} \in K_{n-1}, P_{(s,k_{n-1})} \neq \emptyset\} \subset P$ have been defined at rank $n-1$, and we let, for every $(s, k_n) \in S \times K_n$, such that $P_{(s,k_n)} \neq \emptyset$,

$$g_{(s,k_n)}^n \begin{cases} \text{be equal to } g_{(s,k_{n-1})}^{n-1}, \text{ if there exists } (k_{n-1}, g_{(s,k_{n-1})}^{n-1}) \in K_{n-1} \times G^{n-1} \cap P_{(s,k_n)} \\ \text{be set fixed in } P_{(s,k_n)}, \text{ if } G^{n-1} \cap P_{(s,k_n)} = \emptyset \end{cases} .^2$$

The above selection yields the set $G^n := \{g_{(s,k_n)}^n \in P_{(s,k_n)} : s \in S, k_n \in K_n\} \supset G^{n-1}$.

To complete the induction process, we select $(\omega_0^n, p^n) \in \mathcal{M}^{\varepsilon_0}$ as an equilibrium price of the auxiliary economy, \mathcal{E}^n , presented hereafter (in sub-Section 3.2). Then, we repeat the induction at rank $n+1$, hence, at all ranks. This inductive process yields sequences of prices, $\{(\omega_0^n, p^n)\}_{n \in \mathbb{N}} \in \mathcal{M}^{\varepsilon_0 \mathbb{N}}$, and of non-decreasing price sets, $\{G^n\}_{n \in \mathbb{N}} \in \mathcal{P}(P)^{\mathbb{N}}$, whose union is everywhere dense in P , by construction.

3.2 Auxiliary economies, \mathcal{E}^n

Henceforth, we set $n \in \mathbb{N}^*$ as given. We derive from the set G^n , assumed to be defined by induction, jointly with the past equilibrium price, $(\omega_0^{n-1}, p^{n-1}) \in \mathcal{M}^{\varepsilon_0}$, an auxiliary economy, \mathcal{E}^n , referred to as the n -economy, which is of the type described in [5]. Namely, it is a pure exchange economy, with two period ($t \in \{0, 1\}$), m agents, having incomplete information, and exchanging L goods and J nominal assets, under uncertainty (at $t = 0$) about which state of a finite state space, S^n , will prevail at $t = 1$. Referring to [5], the n -economy's characteristics are as follows:

² Non restrictively (since we can always shift the upper boundary of $P_{(s,k_n)}$), we will assume that each $g_{(s,k_n)}^n \in P_{(s,k_n)}$ is chosen in the interior of $P_{(s,k_n)} \neq \emptyset$, to insure that $\pi_i^{\omega_0}(P_{(s,k_n)}) > 0$, for every $\omega_0 \in \mathcal{M}_0$. This will serve later.

- The information structure is (\tilde{S}_i^n) , where $\tilde{S}_i^n := S \cup S_i^n := S \cup \{i\} \times G^n$, for each $i \in I$. The pooled information set (or the set of states which may prevail tomorrow) is, hence, $S = \cap_{i \in I} \tilde{S}_i^n$. For each $i \in I$, the set $S_i^n := \{i\} \times G^n$ (henceforth identified to G^n) consists of purely formal states, none of which will prevail tomorrow. The formal state space of the n -economy, namely, $S^n = \cup_{i \in I} \tilde{S}_i^n$, is henceforth identified to $S \cup G^n$ in the following definitions, for the sake of simpler notations.
- The $S^n \times J$ payoff matrix, $V^n := (V^n(s^n))_{s^n \in S^n} := (v_j^n(s^n))_{j \in J, s^n \in S^n}$, is defined by $v_j^n(s) := v_j([s, p_s^{-1}])$ and $v_j^n(s^n) := v_j(s^n)$, for each $(j, s, s^n) \in \{1, \dots, J\} \times S \times G^n$. Thus, the financial structure is purely nominal by construction.
- In each formal state, $s^n := [s, p_s] \in S_i^n \cong G^n$, the generic agent $i \in I$ expects with certainty that price $p_s \in \mathbb{R}_{++}^L$ will obtain (if state $s^n := [s, p_s]$ prevails).
- In each realizable state, $s \in S$, the generic agent $i \in I$ has perfect foresight, i.e., anticipates with certainty the true price on the state- s spot market.
- The generic i^{th} agent's endowment, $e_i^n \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^{\tilde{S}_i^n} \cong \mathbb{R}_{++}^L \times \mathbb{R}_{++}^{S^n}$, is defined by $e_i^n(s) := e_i(s)$, for each $s \in S'$, and $e_i^n(s^n) := e_i(s)$, for each $s^n := [s, p_s] \in S_i^n \cong G^n$.
- For every collection, $(\omega_0 := [p_0, q], p := ([s, p_s])_{s \in S}) \in \mathcal{M}$, representing the true prices on all (ex ante) possible spot markets, the generic agent $i \in I$ has the following consumption set, X^n , budget set, $B_i^n(\omega_0, p)$, and utility function, u_i^n :

$$\begin{aligned}
X^n &:= \mathbb{R}_+^L \times \mathbb{R}_+^{\tilde{S}_i^n} \cong \mathbb{R}_+^L \times \mathbb{R}_+^{S^n}; \\
B_i^n(\omega_0, p) &:= \{[x, z] \in X^n \times \mathbb{R}^J : \left[\begin{array}{l} p_0 \cdot (x(0) - e_i(0)) \leq -q \cdot z \\ p_s \cdot (x(s) - e_i(s)) \leq V^n(s) \cdot z, \quad \forall s \in S \\ \delta_s \cdot (x(s^n) - e_i^n(s^n)) \leq V^n(s^n) \cdot z, \quad \forall s^n := [s, \delta_s] \in S_i^n \cong G^n \end{array} \right. \}; \\
u_i^n : x \in X^n &\mapsto u_i^n(x) := \sum_{s^n \in S^n} \pi_i^n(s^n) u_i(x(0), x(s^n)),
\end{aligned}$$

where $\pi_i^n(s^n) := \pi_i^{\omega_0^{n-1}}(P_{(s,k_n)})$, for all $(s, k_n, s^n) \in S \times K_n \times (G^n \cap P_{(s,k_n)})$, is the probability of the set $P_{(s,k_n)}$, along the belief $\pi_i^{\omega_0^{n-1}} := \Pi_i(\omega_0^{n-1}) \in \mathcal{B}$, and $\pi_i^n(s) := \frac{1}{2^n}$, for all $s \in S$.

A Corollary of Theorem 1 in [5] and Lemma 1 above yields Lemma 2 hereafter.

Lemma 2 *The generic n -economy admits an equilibrium, that is, a collection of prices, $(\omega_0, p := ([s, p_s])) \in \mathcal{M}$, and strategies, $([x_i, z_i]) \in \Pi_{i=0}^m B_i^n(\omega_0, p)$, such that:*

$$(a) \forall i \in I, [x_i, z_i] \in \arg \max_{[x, z] \in B_i^n(\omega_0, p)} u_i^n(x);$$

$$(b) \forall s \in S', \sum_{i=1}^m (x_i(s) - e_i(s)) = 0;$$

$$(c) \sum_{i=1}^m z_i = 0.$$

That equilibrium, $\mathcal{C} := ((\omega_0, p), ([x_i, z_i])) \in \mathcal{M} \times \Pi_{i=0}^m B_i^n(\omega_0, p)$, satisfies the relations $(\omega_0, p) \in \mathcal{M}^{\varepsilon_0}$ and $x_i(s) \in [0, e]^L$, for each $(i, s) \in I \times S'$, along Lemma 1.

Proof see the Appendix. □

Along Lemma 2, we set as given an equilibrium of the n -economy, namely:

$$\boxed{\mathcal{C}^n := (\omega_0^n := (p_0^n, q^n), p^n := ([s, p_s^n]), ([x_i^n, z_i^n])) \in \mathcal{M}^{\varepsilon_0} \times \Pi_{i=1}^m B_i^n(\omega_0^n, p^n)},^3$$

henceforth fixed and referred to as the n -equilibrium, which meets the Conditions of Lemma 2. The equilibrium price, $(\omega_0^n, p^n) \in \mathcal{M}^{\varepsilon_0}$, permits to pursue the induction of sub-Section 3.1, at rank $n + 1$, hence, at all ranks.

From Lemma 2, we will assume costlessly that the sequences $\{(\omega_0^n, p^n)\}_{n \in \mathbb{N}^*}$ and $\{x_i^n(s)\}_{n \in \mathbb{N}^*}$, for each $(i, s) \in I \times S'$, converge, respectively, to (ω_0^*, p^*) in the compact

³ Let $Z_n^0 := \{z \in \mathbb{R}^J : V^n(s^n) \cdot z = 0 : \forall s^n \in S^n\}$ be a vector space and Z_n^\perp be its orthogonal complement. Non restrictively, at each step $n \in \mathbb{N}^*$, we let $(z_i^n) \in Z_n^{\perp m}$. In the Appendix (see proof of Lemma 3-(ii) below), this choice will make clear why Hart's (1975) typical problem of a fall in rank of the span at the equilibrium price, which may take place and prevent equilibrium to exist under perfect foresight, cannot occur in our model.

set $\mathcal{M}^{\varepsilon_0}$, and $x_i^*(s)$, in the compact set $[0, e]^L$, such that $\sum_{i \in I} (x_i^*(s) - e_i(s)) = 0$. The following Lemma will serve to prove Theorem 1 in the next sub-Section.

Lemma 3 *Let $G^\infty := \cup_{n \in \mathbb{N}^*} G^n$ and, for every $(\omega, n) \in P \times \mathbb{N}^*$, let $\arg^n(\omega) \in G^n$ be uniquely defined by $(\omega, \arg^n(\omega)) \in P_{(s, k_n)}^2$, for some $(s, k_n) \in S \times K_n$. For every tuple $(i, \omega_0 := [p_0, q], \omega := [s, \delta_s], z) \in I \times \mathcal{M}_0 \times P \times \mathbb{R}^J$, let $B_i(\omega_0, z) := \{y \in \mathbb{R}_+^L : p_0 \cdot (y - e_i(0)) \leq -q \cdot z\}$ and $B_i(\omega, z) := \{y \in \mathbb{R}_+^L : \delta_s \cdot (y - e_i(s)) \leq V(\omega) \cdot z\}$. Then, the following Assertions hold:*

- (i) $G^n \subset G^{n+1}$, $\overline{G}^\infty = P$, $\omega = \lim_{n \rightarrow \infty} \arg^n(\omega)$, $[s, p_s^n] \in P$, $[s, p_s^*] \in P$, $\forall (\omega, n, s) \in P \times \mathbb{N} \times S$;
- (ii) $\exists r \in \mathbb{R}_{++} : \forall n \in \mathbb{N}^*, \sum_{i=1}^m \|z_i^n\| < r$; so, we assume there exists $(z_i^*) = \lim_{n \rightarrow \infty} (z_i^n) \in \mathcal{Z}$;
- (iii) $\exists \gamma \in \mathbb{R}_{++} : \forall (i, n, s^n) \in I \times \mathbb{N}^* \times \{0\} \cup S^n$, $\|x_i^n(s^n)\| \leq \gamma$;
- (iv) $\forall (i, s) \in I \times S$, $x_i^*(s) \in \arg \max_{y \in B_i([s, p_s^*], z_i^*)} u_i(x_i^*(0), y)$, and we let $y_i([s, p_s^*]) := x_i^*(s)$;
- (v) For every $i \in I$, the correspondence $\omega \in P \mapsto \arg \max_{y \in B_i(\omega, z_i^*)} u_i(x_i^*(0), y)$ is upper-semi continuous with non-empty values;
- (vi) For all $(s, \omega := [s, \delta_s]) \in S \times P \setminus \{[s, p_s^*]\}$, we set as given $y_i(\omega) \in \arg \max_{y \in B_i(\omega, z_i^*)} u_i(x_i^*(0), y)$.
The selection $y_i : \omega \in P \mapsto y_i(\omega)$ (defined from (iv) & above) satisfies $y_i^* := [x_i^*(0), y_i] \in Y_i$;
- (vii) as defined from (vi), for every $i \in I$, $u_i^{\omega^*}(y_i^*) = \lim_{n \rightarrow \infty} u_i^n(x_i^n) \in \mathbb{R}_+$;
- (viii) as defined from (vi), $(y_i^*) \in \mathcal{A}(p^*)$.

Proof see the Appendix. □

3.3 An equilibrium of the initial economy, \mathcal{E}_Π

We can prove Theorem 1, via the following Claim.

Claim 1 *The collection of prices, $(\omega_0^*, p^*) = \lim_{n \rightarrow \infty} (\omega_0^n, p^n) \in \mathcal{M}^{\varepsilon_0}$, portfolios, $(z_i^*) = \lim_{n \rightarrow \infty} (z_i^n) \in \mathcal{Z}$, and allocation, $(y_i^*) \in Y := \Pi_{i \in I} Y_i$, along Lemma 3, defines a sequential equilibrium of the economy \mathcal{E}_Π .*

Proof Let $\mathcal{C}^* := ((\omega_0^*, p^*), ([y_i^*, z_i^*]))$ be defined from Claim 1 and use the notations

of Lemma 3. From Lemma 3-(i)-(ii)-(viii), \mathcal{C}^* meets Conditions (a)-(c)-(d) of the above Definition 2 of equilibrium. Hence, Claim 1 will be proved if we show that $[y_i^*, z_i^*] \in B_i(\omega_0^*)$, for each $i \in I$, and that \mathcal{C}^* meets Condition (b) of Definition 2.

Let $i \in I$ be given. From the definition of \mathcal{C}^n , the relations $p_0^n \cdot (x_i^n(0) - e_i(0)) \leq -q^n \cdot z_i^n$ hold, for each $n \in \mathbb{N}^*$, and yield in the limit: $p_0^* \cdot (y_i^*(0) - e_i(0)) := p_0^* \cdot (x_i^*(0) - e_i(0)) \leq -q^* \cdot z_i^*$. The relations $\delta_s \cdot (y_i^*([s, \delta_s]) - e_i(s)) \leq V([s, \delta_s]) \cdot z_i^*$ hold, for all $(s, [s, \delta_s]) \in S \times P$, from Lemma 3-(iv)-(vi), and imply, from Lemma 3-(vi) and above: $[y_i^*, z_i^*] \in \Pi_{i=1}^m B_i(\omega_0^*)$.

Assume, by contraposition, that \mathcal{C}^* fails to meet Definition 2-(b), e.g., there exist $[y_1, z_1] \in B_1(\omega_0^*)$ and $\varepsilon \in \mathbb{R}_{++}$, henceforth set as given, such that:

$$(I) \quad \varepsilon + u_1^{\omega_0^*}(y_1^*) < u_1^{\omega_0^*}(y_1).$$

We may assume that there exists $\eta \in \mathbb{R}_{++}$, such that:

$$(II) \quad y_1^l(\omega) \geq \eta, \text{ for every } (\omega, l) \in \{0\} \cup P \times \mathcal{L}.$$

If not, for every $\alpha \in [0, 1]$, we define the strategy $[y_1^\alpha, z_1^\alpha] := [(1 - \alpha)y_1 + \alpha e_1, (1 - \alpha)z_1]$, which belongs to $B_1(\omega_0^*)$, a convex set. From Assumption A1, the strategy $[y_1^\alpha, z_1^\alpha]$ meets relations (II) whenever $\alpha > 0$. Moreover, from relation (I) and the uniform continuity of $(\alpha, \omega) \in [0, 1] \times P \mapsto u_1(y_1^\alpha(0), y_1^\alpha(\omega))$ on a compact set (which holds from Assumption A2 and the relation $y_1 \in Y_1$), the strategy $[y_1^\alpha, z_1^\alpha]$ also meets relation (I), for every $\alpha > 0$, small enough. So, we may assume relations (II).

We let the reader check, as immediate from the relations (II), $[y_1, z_1] \in B_1(\omega_0^*)$ and $(\omega_0^*, p^*) \in \mathcal{M}^{\varepsilon_0}$, Assumption A2, same continuity arguments as above (& that of the scalar product), that we may also assume there exists $\gamma \in \mathbb{R}_{++}$, such that:

$$(III) \quad p_0^* \cdot (y_1(0) - e_1(0)) \leq \gamma - q^* \cdot z_1 \text{ and } \delta_s \cdot (y_1([s, \delta_s]) - e_1(s)) \leq \gamma + V([s, \delta_s]) \cdot z_1, \forall [s, \delta_s] \in P.$$

Finally, from relations (III), the continuity of V and of the scalar product, the relation $(\omega_0^*, p^*) = \lim_{n \rightarrow \infty} (\omega_0^n, p^n)$ and Lemma 3-(i)-(ii), there exists $N_1 \in \mathbb{N}$, such that:

$$(IV) \quad \left\{ \begin{array}{l} p_0^n \cdot (y_1(0) - e_1(0)) \leq -q^n \cdot z_1 \\ p_s^n \cdot (y_1([s, p_s^n]) - e_1(s)) \leq V^n(s) \cdot z_1, \quad \forall s \in S \\ \delta_s \cdot (y_1(s^n) - e_1^n(s^n)) \leq V^n(s^n) \cdot z_1, \quad \forall s^n := [s, \delta_s] \in S_1^n \cong G^n \end{array} \right\}, \text{ for all } n \geq N_1.$$

Along relations (IV) and Lemma 3-(i), for each $n \geq N_1$, we let $[y_1^n, z_1] \in B_1^n(\omega_0^n, p^n)$ be the strategy defined by $y_1^n(0) := y_1(0)$, $y_1^n(s) := y_1([s, p_s^n])$, for every $s \in S$, and $y_1^n(s^n) := y_1(s^n)$, for every $s^n \in S_1^n \cong G^n$, and recall that:

- $u_1^{\omega_0^*}(y_1) := \int_{\omega \in P} u_1(y_1(0), y_1(\omega)) d\pi_1^{\omega_0^*}(\omega)$ and $u_1^{\omega_0^*}(y_1^*) := \int_{\omega \in P} u_1(y_1^*(0), y_1^*(\omega)) d\pi_1^{\omega_0^*}(\omega)$;
- $u_1^n(x_1^n) - \sum_{s \in S} \frac{u_1^n(x_1^n(0), x_1^n(s))}{2^n} := \sum_{\omega \in G^n} u_1(x_1^n(0), x_1^n(\omega)) \pi_1^n(\omega)$;
- $u_1^n(y_1^n) - \sum_{s \in S} \frac{u_1^n(y_1^n(0), y_1^n(s))}{2^n} := \sum_{\omega \in G^n} u_1(y_1^n(0), y_1^n(\omega)) \pi_1^n(\omega)$.

Then, from the above definitions, the uniform continuity of $\omega \in P \mapsto u_1(y_1(0), y_1(\omega))$ on a compact set and Condition (d) of Definition 1 applied to $y_1 \in Y_1$, it is immediate that there exists $N_2 \geq N_1$ such that the following relations hold, for every $n \geq N_2$:

$$(V) \quad \left\{ \begin{array}{l} |u_1^{\omega_0^*}(y_1) - u_1^n(y_1^n)| \\ < \frac{\varepsilon}{8} + \left| \int_{\omega \in P} u_1(y_1(0), y_1(\omega)) d\pi_1^{\omega_0^*}(\omega) - \sum_{\omega \in G^n} u_1(y_1^n(0), y_1^n(\omega)) \pi_1^n(\omega) \right| \\ < \frac{\varepsilon}{8} + \left| \int_{\omega \in P} u_1(y_1(0), y_1(\omega)) d\pi_1^{\omega_0^*}(\omega) - \int_{\omega \in P} u_1(y_1(0), y_1(\omega)) d\pi_1^{\omega_0^{n-1}}(\omega) \right| \\ + \left| \int_{\omega \in P} u_1(y_1(0), y_1(\omega)) d\pi_1^{\omega_0^{n-1}}(\omega) - \sum_{\omega \in G^n} u_1(y_1^n(0), y_1^n(\omega)) \pi_1^n(\omega) \right| \\ < \frac{\varepsilon}{4} + \left| \int_{\omega \in P} u_1(y_1(0), y_1(\omega)) d\pi_1^{\omega_0^{n-1}}(\omega) - \sum_{\omega \in G^n} u_1(y_1^n(0), y_1^n(\omega)) \pi_1^n(\omega) \right| \\ < \frac{\varepsilon}{2} \end{array} \right\}.$$

From the definition of n -equilibria (for each $n \in \mathbb{N}^*$) and Lemma 3-(vii), there exists $N_3 \geq N_2$, such that:

(VI) $u_1^n(y_1^n) \leq u_1^n(x_1^n) < \frac{\varepsilon}{2} + u_1^{\omega_0^*}(y_1^*)$, for every $n \geq N_3$.

Let $n \geq N_3$ be given. The above Conditions (I)-(V)-(VI) yield, jointly:

$$u_1^{\omega_0^*}(y_1) < \frac{\varepsilon}{2} + u_1^n(y_1^n) < \varepsilon + u_1^{\omega_0^*}(y_1^*) < u_1^{\omega_0^*}(y_1).$$

This contradiction proves that \mathcal{C}^* is a C.F.E., hence, it completes the proof of Claim 1 and of Theorem 1. \square

Appendix: proof of the Lemmas

Lemma 2 *The generic n -economy admits an equilibrium, that is, a collection of prices, $(\omega_0, p := ([s, p_s])) \in \mathcal{M}$, and strategies, $([x_i, z_i]) \in \Pi_{i=0}^m B_i^n(\omega_0, p)$, such that:*

$$(a) \forall i \in I, [x_i, z_i] \in \arg \max_{[x, z] \in B_i^n(\omega_0, p)} u_i^n(x);$$

$$(b) \forall s \in S', \sum_{i=1}^m (x_i(s) - e_i(s)) = 0;$$

$$(c) \sum_{i=1}^m z_i = 0.$$

That equilibrium, $\mathcal{C} := ((\omega_0, p), ([x_i, z_i])) \in \mathcal{M} \times \Pi_{i=0}^m B_i^n(\omega_0, p)$, satisfies the relations $(\omega_0, p) \in \mathcal{M}^{\varepsilon_0}$ and $x_i(s) \in [0, e]^L$, for each $(i, s) \in I \times S'$, along Lemma 1.

Proof Let $n \in \mathbb{N}^*$ be given. The n -economy is, formally, one of the type presented in [5] and admits an equilibrium along Definition 3 and Theorem 1 of [5] and its proof, that is, a collection of (normalized) prices, $(\omega_0, p) \in \mathcal{M}$, and strategies, $([x_i, z_i]) \in \Pi_{i=1}^m B_i^n(\omega_0, p)$, which satisfy Conditions (a)-(b)-(c) of Lemma 2.

Let $\mathcal{C} := (\omega_0, p, ([x_i, z_i]))$ be an equilibrium of \mathcal{E}^n and $(e, \beta, \varepsilon_0) \in \mathbb{R}_{++}^3$ be the bounds of Lemma 1. Then, for each $s \in S'$, the relations $x_i(s) \geq 0$ and $\sum_{i=1}^m (x_i(s) - e_i(s)) = 0$, which hold from Lemma 2-(b), imply $x_i(s) \in [0, e]^L$, for each $i \in I$, whereas the relation $p_s \in \mathbb{R}_{++}^L$ is standard (from Assumption A2 and Lemma 2-(a)). Let $(s, (l, l')) \in S \times \mathcal{L}^2$ be given. We show, first, that the relation $\frac{p_s^l}{p_s^{l'}} \leq \beta$ holds for

$\beta := \sup_{i \in I, (l, l') \in \mathcal{L}^2, (x, y) \in ([0, e]^L)^2} \frac{\partial u_i}{\partial v^l}(x, y) / \frac{\partial u_i}{\partial v^{l'}}(x, y) \in \mathbb{R}$, along Lemma 1. Otherwise, it is easy to check (from Lemma 2-(b) and Assumptions A1-A2) that there exist an agent, $i \in I$, such that $x_i^l(s) > 0$, and a consumption, $y_i \in X^n$, defined by $y_i^{l'}(s) := x_i^{l'}(s) + \frac{\varepsilon}{p_s^{l'}}$ and $y_i^l(s) := x_i^l(s) - \frac{\varepsilon}{p_s^l}$ (for $\varepsilon \in \mathbb{R}_{++}$ small enough), and $y_i^l(s^n) := x_i^l(s^n)$, for every $(s^n, \bar{l}) \in \tilde{S}_i^n \times \mathcal{L} \setminus \{(s, l), (s, l')\}$, such that $[y_i, z_i] \in B_i^n(\omega_0, p)$ and $u_i^n(y_i) > u_i^n(x_i)$. The latter conditions contradict the fact that \mathcal{C} meets Lemma 2-(a). We let the reader check that the joint relations $p_s >> 0$, $\|p_s\| = 1$ and $\frac{p_s^l}{p_s^{l'}} \leq \beta$, which hold from above for all $(s, (l, l')) \in S \times \mathcal{L}^2$, imply $p_s^l \geq \frac{1}{\beta \sqrt{L}}$, for all $(s, l) \in S \times \mathcal{L}$, i.e., $(\omega_0, p) \in \mathcal{M}^{\varepsilon_0}$. \square

Lemma 3 *Let $G^\infty := \cup_{n \in \mathbb{N}^*} G^n$ and, for every $(\omega, n) \in P \times \mathbb{N}^*$, let $\arg^n(\omega) \in G^n$ be uniquely defined by $(\omega, \arg^n(\omega)) \in P_{(s, k_n)}^2$, for some $(s, k_n) \in S \times K_n$. For every tuple $(i, \omega_0 := [p_0, q], \omega := [s, \delta_s], z) \in I \times \mathcal{M}_0 \times P \times \mathbb{R}^J$, let $B_i(\omega_0, z) := \{y \in \mathbb{R}_+^L : p_0 \cdot (y - e_i(0)) \leq -q \cdot z\}$ and $B_i(\omega, z) := \{y \in \mathbb{R}_+^L : \delta_s \cdot (y - e_i(s)) \leq V(\omega) \cdot z\}$. Then, the following Assertions hold:*

- (i) $G^n \subset G^{n+1}$, $\overline{G}^\infty = P$, $\omega = \lim_{n \rightarrow \infty} \arg^n(\omega)$, $[s, p_s^n] \in P$, $[s, p_s^*] \in P$, $\forall (\omega, n, s) \in P \times \mathbb{N} \times S$;
- (ii) $\exists r \in \mathbb{R}_{++} : \forall n \in \mathbb{N}^*, \sum_{i=1}^m \|z_i^n\| < r$; so, we assume there exists $(z_i^*) = \lim_{n \rightarrow \infty} (z_i^n) \in \mathcal{Z}$;
- (iii) $\exists \gamma \in \mathbb{R}_{++} : \forall (i, n, s^n) \in I \times \mathbb{N}^* \times \{0\} \cup S^n$, $\|x_i^n(s^n)\| \leq \gamma$;
- (iv) $\forall (i, s) \in I \times S$, $x_i^*(s) \in \arg \max_{y \in B_i([s, p_s^*], z_i^*)} u_i(x_i^*(0), y)$, and we let $y_i([s, p_s^*]) := x_i^*(s)$;
- (v) For every $i \in I$, the correspondence $\omega \in P \mapsto \arg \max_{y \in B_i(\omega, z_i^*)} u_i(x_i^*(0), y)$ is upper-semi continuous with non-empty values;
- (vi) For all $(s, \omega := [s, \delta_s]) \in S \times P \setminus \{[s, p_s^*]\}$, we set as given $y_i(\omega) \in \arg \max_{y \in B_i(\omega, z_i^*)} u_i(x_i^*(0), y)$. The selection $y_i : \omega \in P \mapsto y_i(\omega)$ (defined from (iv) & above) satisfies $y_i^* := [x_i^*(0), y_i] \in Y_i$;
- (vii) as defined from (vi), for every $i \in I$, $u_i^{\omega_0^*}(y_i^*) = \lim_{n \rightarrow \infty} u_i^n(x_i^n) \in \mathbb{R}_+$;
- (viii) as defined from (vi), $(y_i^*) \in \mathcal{A}(p^*)$.

Proof Assertion (i) is immediate from the definitions (in particular, Condition (d) of Definition 1 and the relation $(\omega_0^n, p^n) \in \mathcal{M}^{\varepsilon_0}$ of the generic n -equilibrium), and from the compactness of P . \square

Assertion (ii). From Definition 1, let $\varepsilon \in]0, 1]$ (which exists) satisfy $P \subset \Omega^\varepsilon$ and let $\alpha := \max_{i \in I} \frac{\|e_i\|}{\varepsilon}$. Then, it follows from the definition of \mathcal{C}^n (for each $n \in \mathbb{N}^*$) that:

$$(I) \quad [(z_i^n) \in \mathcal{Z} \text{ and } V(s^n) \cdot z_i^n \geq -\alpha, \forall (i, n, s^n) \in I \times \mathbb{N}^* \times S^n].$$

We recall from footnote 3 and the definition of V^n that $z_i^n \in Z_n^\perp := (Z_n^0)^\perp$, for each $(i, n) \in I \times \mathbb{N}^*$, where $Z_n^0 := \{z \in \mathbb{R}^J : V(s^n) \cdot z = 0 : \forall s^n \in S^n\}$ is a vector space. The fact that $\{S^n\}_{n \in \mathbb{N}^*}$ is non-decreasing implies that $\{Z_n^\perp\}_{n \in \mathbb{N}^*}$ is non-decreasing in $(\mathbb{R}^J)^m$. Hence, there exists $N_0 \in \mathbb{N}^*$, henceforth fixed, such that $Z_n^\perp = Z_{N_0}^\perp$ for every $n \geq N_0$.

Assume, by contradiction, that Lemma 3-(ii) fails, i.e., there exists an extracted sub-sequence of n -equilibria, $\{\mathcal{C}^{\varphi(n)}\}_{n \in \mathbb{N}^*} := \{(\omega_0^{\varphi(n)}, p^{\varphi(n)}, ([x_i^{\varphi(n)}, z_i^{\varphi(n)}])\}_{n \in \mathbb{N}^*}$, such that $n < \|z_i^{\varphi(n)}\| \leq n + 1$, for each $n \in \mathbb{N}^*$. For each $n \geq N_0$, the strategy collection $([\tilde{z}_i^n, \tilde{z}_i^n]) := (\frac{1}{n}[x_i^{\varphi(n)} + (n-1)e_i^{\varphi(n)}, z_i^{\varphi(n)}])$, belongs to the convex set $\Pi_{i=1}^m B_i^{\varphi(n)}(\omega_0^{\varphi(n)}, \pi_i^{\varphi(n)})$ and, from above, meets the relations $(\tilde{z}_i^n) \in \mathcal{Z} \cap Z_{N_0}^{\perp m}$ and $1 < \|(\tilde{z}_i^n)\| \leq 1 + \frac{1}{n}$. Hence, from Condition (I) and above:

$$(II) \quad [(\tilde{z}_i^n) \in \mathcal{Z} \cap (Z_{N_0}^\perp)^m \text{ and } V(s^n) \cdot \tilde{z}_i^n \geq -\frac{\alpha}{n}, \forall i \in I, \forall s^n \in S^{N_0} \subset S^{\varphi(n)}], \text{ for each } n \geq N_0.$$

From Condition (II), the bounded sequence $\{(\tilde{z}_i^n)\}_{n \geq N_0}$ may be assumed to converge, say to $(\tilde{z}_i^*) \in \mathcal{Z} \cap (Z_{N_0}^\perp)^m$, a closed set, such that $\|(\tilde{z}_i^*)\| = 1$, and, from the continuity of the scalar product:

$$(III) \quad [(\tilde{z}_i^*) \in \mathcal{Z} \cap (Z_{N_0}^\perp)^m \text{ and } V(s^n) \cdot \tilde{z}_i^* \geq 0, \forall (i, s^n) \in I \times S^{N_0}].$$

From ([3], Propositions 2.1 & 3.1, pp. 398 & 401), that Condition (III) implies that $(\tilde{z}_i^*) \in (Z_{N_0}^0)^m \cap (Z_{N_0}^\perp)^m = \{0\}$, which contradicts the above relation, $\|(\tilde{z}_i^*)\| = 1$. \square

Assertion (iii) Let $W := \sup_{[s, p_s] \in P} \|V([s, p_s])\| \in \mathbb{R}_+$ (since V is continuous and P is compact), $e := \max_{(s, l) \in S' \times \mathcal{L}} \sum_{i=1}^m e_i^l(s)$, $\varepsilon := \inf_{[s, p_s] \in P, l \in \mathcal{L}} p_s^l \in]0, \varepsilon_0]$, along Definition

1, and $Z := \sup_{n \in \mathbb{N}^*} \|(z_i^n)\| \in \mathbb{R}_+$, along Lemma 3-(ii), be given. Then, for every $(i, n, s^n) \in I \times \mathbb{N}^* \times S^n$, Lemma 3-(i) and the relation $[x_i^n, z_i^n] \in B_1(\omega_0^n, p^n)$ imply:

$$\delta_s^n \cdot (x_i^n(s^n) - e_i(s)) = \sum_{l=1}^L \delta_s^{nl} \cdot (x_i^{nl}(s^n) - e_i^l(s)) \leq WZ;$$

$$\delta_s^{nl} \cdot (x_i^{nl}(s^n) - e_i^l(s)) \leq WZ + e;$$

$$x_i^{nl}(s^n) \leq \frac{WZ+e}{\varepsilon} + e;$$

$$\|x_i^n(s^n)\| \leq \gamma := L \frac{WZ+e}{\varepsilon} + Le.$$

Hence, from Lemma 2, $\|x_i^n(s^n)\| \leq \gamma := L \frac{WZ+e}{\varepsilon} + Le$, for all $(n, s^n) \in \mathbb{N}^* \times \{0\} \cup S^n$. \square

Assertion (iv) Let $(i, s) \in I \times S$ be given.

For every $(i, n, \omega_0 := [p_0, q], \omega := [s, \delta_s], \omega', z) \in I \times \mathbb{N}^* \times \mathcal{M}_0 \times P \times P \times \mathbb{R}^J$, we recall and let:

$$B_i(\omega_0, z) := \{y \in \mathbb{R}_+^L : p_0 \cdot (y - e_i(0)) \leq -q \cdot z\};$$

$$B_i(\omega, z) := \{y \in \mathbb{R}_+^L : \delta_s \cdot (y - e_i(s)) \leq V(\omega) \cdot z\};$$

$$B'_i(\omega, \omega', z) := \{y \in \mathbb{R}_+^L : \delta_s \cdot (y - e_i(s)) \leq V(\omega') \cdot z\}.$$

For each $n \in \mathbb{N}^*$, the fact that \mathcal{C}^n is a n -equilibrium implies:

$$(I) \quad x_i^n(s) \in \arg \max_{y \in B'_i([s, p_s^n], [s, p_s^{n-1}], z_i^n)} u_i(x_i^n(0), y).$$

As a standard result (see, e.g., [6], 4, p. 19), the correspondence $(x, \omega, \omega', z) \mapsto \arg \max_{y \in B_i(\omega, \omega', z)} u_i(x, y)$ is upper semi-continuous, from the continuity of the mapping, u_i , and correspondence, B'_i , whereas, from Lemma 3-(ii) and above, $(x_i^*(0), x_i^*(s), p_s^*, z_i^*) = \lim_{n \rightarrow \infty} (x_i^n(0), x_i^n(s), p_s^n, z_i^n)$. Hence, the relations of Condition (I) pass to the limit, which yields: $x_i^*(s) \in \arg \max_{y \in B_i([s, p_s^*], z_i^*)} u_i(x_i^*(0), y)$. \square

Assertion (v) For all $(i, \omega, n) \in I \times P \times \mathbb{N}^*$, the fact that \mathcal{C}^n is a n -equilibrium yields:

$$(I) \quad x_i^n(\arg^n(\omega)) \in \arg \max_{y \in B_i(\arg^n(\omega), z_i^n)} u_i(x_i^n(0), y).$$

From Lemma 3-(i)-(ii), the relation $(\omega, x_i^*(0), z_i^*) = \lim_{n \rightarrow \infty} (\arg^n(\omega), x_i^n(0), z_i^n)$ holds, whereas, from ([6], 4, p. 19) and the continuity of u_i and B_i , the correspondence $(x, \omega, z) \mapsto \arg \max_{y \in B_i(\omega, z)} u_i(x, y)$ is upper semi-continuous, for every $(i, \omega) \in I \times P$. Hence, passing to the limit into Condition (I) yields: $\arg \max_{y \in B_i(\omega, z_i^*)} u_i(x_i^*(0), y) \neq \emptyset$ and $\arg \max_{y \in B_i(\omega, z_i^*)} u_i(x_i^*(0), y)$ is upper semi-continuous, for every $(i, \omega) \in I \times P$. \square

Assertion (vi) Let $i \in I$ be given and y_i be defined along Lemma 3-(vi), that is, for every $\omega \in P \setminus \cup_{s \in S} [s, p_s^*]$, we let $y_i(\omega)$ be a selection $y_i(\omega) \in \arg \max_{y \in B_i(\omega, z_i^*)} u_i(x_i^*(0), y)$ and, for each $s \in S$, we let $y_i([s, p_s^*]) := x_i^*(s)$. From Lemma 3-(i)-(iv)-(v), the mapping $\omega \in P \mapsto y_i(\omega) \in \mathbb{R}_+^L$ is perfectly defined and such that $y_i(\omega) \in \arg \max_{y \in B_i(\omega, z_i^*)} u_i(x_i^*(0), y)$, for every $\omega \in P$. Moreover, since u_i and B_i are continuous, it follows from ([6], 4, p. 19), that the mapping $\omega \in P \mapsto u_i(x_i^*(0), y_i(\omega))$ is continuous, i.e., $y_i^* := [x_i^*(0), y_i] \in Y_i$. \square

Assertion (vii) Let $i \in I$ and $y_i^* := [x_i^*(0), y_i] \in Y_i$ be given, along Lemma 3-(vi).

As above, from the continuity of u_i and B_i , and from ([6], 4, p. 19), the mapping $[y_0, \omega, z] \in \mathbb{R}_+^L \times P \times \mathbb{R}^J \mapsto U_i(y_0, \omega, z) = \max_{y \in B_i(\omega, z)} u_i(y_0, y)$ is continuous (on its domain), whereas the relation $u_i(x_i^n(0), x_i^n(\omega^n)) = U_i(x_i^n(0), \omega^n, z_i^n)$ holds, for every $\omega^n \in G^n$, from the fact that C^n is a n -equilibrium, and $u_i(x_i^*(0), y_i(\omega)) = U_i(x_i^*(0), \omega, z_i^*)$ holds, for every $\omega \in P$, from the definition of y_i .

From Lemma 3-(i), for every $\omega \in P$, the sequence $\{\omega^n\} := \{\arg^n(\omega) \in G^n\}$, converges (uniformly on P) to ω . From above, the continuity of U_i and Lemma 3-(ii), the relations $u_i(x_i^*(0), y_i(\omega)) = U_i(x_i^*(0), \omega, z_i^*) = \lim_{n \rightarrow \infty} U_i(x_i^n(0), \omega^n, z_i^n) = \lim_{n \rightarrow \infty} u_i(x_i^n(0), x_i^n(\omega^n))$ hold, for every $\omega \in P$. These relations yield, from the uniform continuity of U_i on a compact set, and the uniform convergence of $\{(x_i^n(0), \omega^n, z_i^n)\}$ to $(x_i^*(0), \omega, z_i^*)$:

$$(I) \quad \forall \varepsilon > 0, \exists n_\varepsilon^1 \in \mathbb{N}^* : \forall n > n_\varepsilon^1, \forall \omega \in P, \|u_i(x_i^*(0), y_i(\omega)) - u_i(x_i^n(0), x_i^n(\omega^n))\| < \varepsilon.$$

For every $n \in \mathbb{N}^*$, we let $y_i^n : P \rightarrow \mathbb{R}_+^L$ be defined (from Lemma 3-(i)) by:

- $y_i^n(0) := x_i^n(0) \in \mathbb{R}_+^L$ and $y_i^n(\omega) := x_i^n(\arg^n(\omega)) := x_i^n(\omega^n) \in \mathbb{R}_+^L$, for every $\omega \in P$;

We recall the following definitions (for each $n \in \mathbb{N}^*$):

- $u_i^{\omega_0^*}(y_i^*) := \int_{\omega \in P} u_i(y_i^*(0), y_i(\omega)) d\pi_i^{\omega_0^*}(\omega)$;
- $u_i^n(x_i^n) - \sum_{s \in S} \frac{u_i(x_i^n(0), x_i^n(s))}{2^n} := \sum_{\omega \in G^n} u_i(x_i^n(0), x_i^n(\omega)) \pi_i^n(\omega) = \int_{\omega \in P} u_i(y_i^n(0), y_i^n(\omega)) d\pi_i^{\omega_0^{n-1}}(\omega)$.

Let $\varepsilon > 0$ be given. From Assumption A2, Lemma 3-(iii)-(vi), Condition (I) above and Definition 1-(a), the following relations hold, in steps:

$$\begin{aligned}
(II) \quad & \exists n_\varepsilon^2 > n_\varepsilon^1 : \forall n > n_\varepsilon^2, |u_i^{\omega_0^*}(y_i^*) - u_i^n(x_i^n)| \\
& < \varepsilon + \left| \int_{\omega \in P} u_i(y_i^*(0), y_i(\omega)) d\pi_i^{\omega_0^*}(\omega) - \int_{\omega \in P} u_i(y_i^n(0), y_i^n(\omega)) d\pi_i^{\omega_0^{n-1}}(\omega) \right| \\
& \leq \varepsilon + \left| \int_{\omega \in P} u_i(y_i^*(0), y_i(\omega)) d\pi_i^{\omega_0^*}(\omega) - \int_{\omega \in P} u_i(y_i^*(0), y_i(\omega)) d\pi_i^{\omega_0^{n-1}}(\omega) \right| \\
& + \left| \int_{\omega \in P} [u_i(y_i^*(0), y_i(\omega)) - u_i(y_i^n(0), y_i^n(\omega))] d\pi_i^{\omega_0^{n-1}}(\omega) \right| \\
& \leq 2\varepsilon + \left| \int_{\omega \in P} [u_i(x_i^*(0), y_i(\omega)) - u_i(x_i^n(0), x_i^n(\omega^n))] d\pi_i^{\omega_0^{n-1}}(\omega) \right| \leq 3\varepsilon.
\end{aligned}$$

That is, $u_i^{\omega_0^*}(y_i^*) = \lim_{n \rightarrow \infty} u_i^n(x_i^n)$. □

Assertion (viii) From the definition of \mathcal{C}^n , the relations $\sum_{i=1}^m (x_i^n(s) - e_i(s)) = 0$ hold, for every $(n, s) \in \mathbb{N}^* \times S'$, and, passing to the limit ($n \rightarrow \infty$), yield: $\sum_{i=1}^m (x_i^*(s) - e_i(s)) = 0$, for each $s \in S'$. The latter relations imply, from Lemma 3-(vi): $(y_i^*) \in \mathcal{A}(p^*)$. □

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